CIRCLES

STANDARD FORMS OF EQUATION OF A CIRCLE

General Form

The general equation of a circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$, where g,f,c are constants.

For this circle, Centre = $(-g, -f)$, Radius = $\sqrt{g^2 + f^2 - c}$

Central Form

If (h, k) be the centre and 'a' be the radius of a circle, then its equation is

$$(x-h)^2 + (y-k)^2 = a^2$$

Note : If the centre is origin, then the eqation of the circle is $x^2 + y^2 = r^2$ (simplest form)

Diameter Form

If (x_1, y_1) and (x_2, y_2) are end points of a diameter of a circle then its equation is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Parametric Equation of a Circle

(a) The parametric equations of a circle $x^2 + y^2 = a^2$ are

$$x = a\cos\theta, \; y = a\sin\theta.$$

Hence parametric coordinates of any point lying on the circle $x^2 + y^2 = a^2$ are

$$(a\cos\theta, a\sin\theta)$$

(b) The parametric equations of the circle $(x - h)^2 + (y - k)^2 = a^2$ are

$$x = h + a\cos\theta, \; y = k + a\sin\theta.$$

Hence parametric coordinates of any point lying on the circle are

$$(h + a\cos\theta, k + a\sin\theta)$$

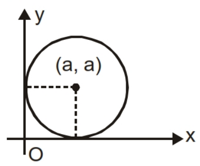
(c) Parametric equations of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$x = -g + \sqrt{g^2 + f^2 - c}\cos\theta, \quad y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$$

EQUATION OF A CIRCLE IN SOME SPECIAL CASES

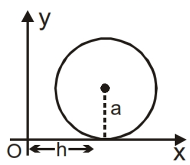
(i) Which touches both axes : The equation of a circle with radius 'a' touching both coordinate axes is given by

$$(x \pm a)^2 + (y \pm a)^2 = a^2$$



(ii) Which touches x-axis : The equation of a circle with radius 'a' touching x-axis at a distance h from the origin is

$$(x - h)^2 + (y - a)^2 = a^2$$



Note : The equation of a circle with radius 'a' touching x-axis at the origin is

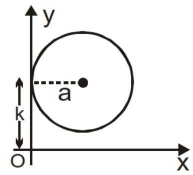
$$x^2 + (y \pm a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 \pm 2ay = 0$$

(iii) Which touches y-axis : The equation of a circle with radius 'a' touching y-axis at a distance k from the origin is

$$(x - a)^2 + (y - k)^2 = a^2$$

Note : The equation of a circle with radius 'a' touching y-axis at the origin is
$(x \pm a)^2 + y^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 \pm 2ax = 0$



INTERCEPT ON A LINE OR LENGTH OF A CHORD

The length of a chord AB of a circle (or the intercept made by a circle on a line) is given by

$AB = 2\sqrt{a^2 - p^2}$

where 'a' is the radius of the circle and 'p' is the length of the perpendicular from its centre on the chord.
In particular the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts

(i) intercept on x-axis $= 2\sqrt{g^2 - c}$

(ii) intercept on y-axis $= 2\sqrt{f^2 - c}$

POSITION OF A POINT AND LINE WITH RESPECT TO A CIRCLE

Position of a point

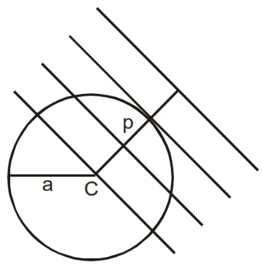
A point (x_1, y_1) lies outside, on or inside a circle $S\ ^{o}\ x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1\ ^{o}\ x_1^{\ 2} + y_1^{\ 2} + 2gx_1 + 2fy_1 + c$ is positive, zero or negative. So

- $S_1 > 0 \quad \Rightarrow \quad (x_1, y_1)$ is outside the circle
- $S_1 = 0 \quad \Rightarrow \quad (x_1, y_1)$ is on the circle
- $S_1 < 0 \quad \Rightarrow \quad (x_1, y_1)$ is inside the circle

Position of a line

Let $L = 0$ be a line and $S = 0$ be a circle. If 'a' be the radius of the circle and 'p' be the length of the perpendicular from its centre on the line, then

- $p > a \Rightarrow$ line is outside the circle
- $p = a \Rightarrow$ line touches the circle
- $p < a \Rightarrow$ line is a chord of the circle
- $p = 0 \Rightarrow$ line is a diameter of the circle



CONDITION OF TANGENCY

A line $L = 0$ touches the circle $S = 0$, if length of perpendicular drawn from the centre of the circle to the line is equal to radius of the circle i.e., $p = r$. This is the condition of tangency for the line $L = 0$.

The line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1+m^2}$

Thus, for every value of m, the line $y = mx \pm a\sqrt{1+m^2}$

is a tangent of the circle $x^2 + y^2 = a^2$ and its point of contact is $\left(\dfrac{\mp am}{\sqrt{1+m^2}}, \dfrac{\mp a}{\sqrt{1+m^2}}\right)$

Note :

- If $a^2(1 + m^2) - c^2 > 0$ line will meet the circle at real and different points.
- If $c^2 = a^2(1 + m^2)$ line will touch the circle.
- If $a^2(1 + m^2) - c^2 > 0$ line will meet circle at two imaginary points.

EQUATION OF THE TANGENT AND NORMALS AT A POINT

Equation of a tangent

The equation of the tangent at a point (x_1, y_1) of a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 = T$

Equation of the Normal

The equation of the normal at the point (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is $\Rightarrow \dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

Note : For the circle $x^2 + y^2 = a^2$ it becomes $\dfrac{x}{x_1} = \dfrac{y}{y_1}$

Length of the tangent

The length of the tangent drawn from a point $P(x_1, y_1)$ to the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $PQ = PR = \sqrt{S_1}$

where $S_1 = x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c$

Also area of the quadrilateral PQCR $= r\sqrt{S_1}$ and angle between tangents PQ and PR i.e.

$$\angle QPR = 2\tan^{-1}\frac{r}{\sqrt{S_1}}$$

Equation of the pair of tangents

From a given point $P(x_1, y_1)$, two tangents PQ and PR can be drawn to a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$. Their combined equation is given by $SS_1 = T^2$

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents of a circle is called the director circle of that circle.

The equation of the director circle of $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$

It may be easily seen that

· Centre of the director circle = centre of the given circle.

· Radius of the director circle = $\sqrt{2}$ (radius of the given circle)

CHORD OF CONTACT

The chord joining the two points of contact of tangents to a circle drawn from any point P is called chord of contact of P with respect to the given circle.

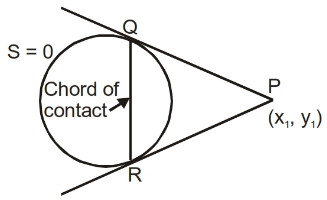
Let the given point is $P(x_1, y_1)$ and the circle is $S = 0$ then equation of the chord of contact is

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note:

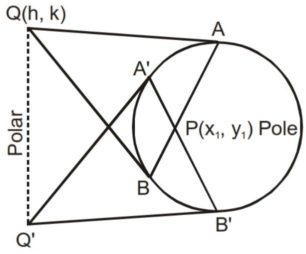
· Length of chord of contact = $2\sqrt{r^2 - p^2}$

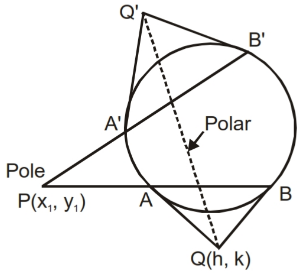
· Area of triangle PQR = $\dfrac{a(x_1^2 + y_1^2 - a^2)^{3/2}}{x_1^2 + y_1^2}$



POLE & POLAR

Let $P(x_1, y_1)$ be any point inside or outside the circle. Draw chords AB and A' B' pasing through P. If tangent to the circle at A and B meet at Q (h, k), then locus of Q is called polar of P.w.r.t. circle and P is called the pole and if tangent to the circle at A' and B' meet at Q', then the straigt line QQ' is polar with P' as its pole.





Equation of polar

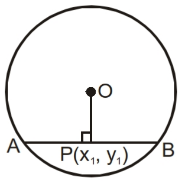
- Equation of polar of the pole $P(x_1, y_1)$ w.r.t. $x^2 + y^2 = a^2$ is
  $$xx_1 + yy_1 = a^2$$
- Equation of polar of the pole (x_1, y_1) w.r.t. circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is
  $$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Coordinates of pole

- Pole of polar $Ax + By + C = 0$ w.r.t. circle $x^2 + y^2 = a^2$ is $\left(-\frac{Aa^2}{C}, -\frac{Ba^2}{C}\right)$
- Pole of polar $Ax + By + C = 0$ with respect to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by the equation
  $$\frac{x_1 + g}{A} = \frac{y_1 + f}{B} = \frac{gx_1 + fy_1 + c}{C}$$

Conjugate points and Conjugate lines

(i) Conjugate points :- Two points are called conjugate points with respect to a circle if each point lies on the polar of the other point with respect to the same circle.



(ii) Conjugate lines :- Two lines are called conjugate lines with respect to a circle if the pole of each line lies on the other line.

CHORD WITH A GIVEN MIDDLE POINT

The equation of the chord of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ whose mid-point is (x_1, y_1) is given by $T = S_1$

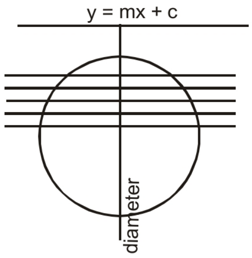
$$\text{i.e. } xx_1 + yy_1 + g(x + x_1)\ f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$$

In particular, the equation of the chord of the circle $x^2 + y^2 = a^2$ whose middel point is (x_1, y_1) is $T = S_1$

$$\text{i.e., } xx_1 + yy_1 = x_1^2 + y_1^2$$

13. DIAMETER OF A CIRCLE

The locus of middle points of a system of parallel chords of a circle is called the diameter of that circle. The diameter of the circle $x^2 + y^2 = r^2$ corresponding to the system of parallel chords $y = mx + c$ is $x + my = 0$

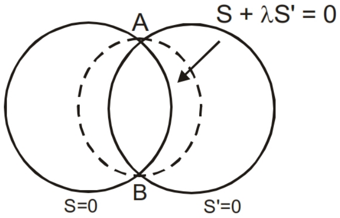


Note : · Every Diameter passes through the centre of the circle.

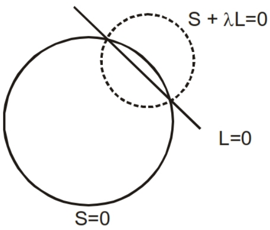
· A diameter is perpendicular to the system of parallel chords.

FAMILY OF CIRCLES

If S = 0 and S' = 0 are two intersecting circles, the $S + \lambda S' = 0$ ($\lambda \neq -1$) represents family of circles passing throguh intersection points of S = 0 and S' = 0 (λ being parameter)



If S = 0 and L = 0 represent a circle and a line, then $S + \lambda L = 0$ represent family of circles passing through intersection points of circle S= 0 and line L = 0 (λ being parameter)



COMMON CHORD OF TWO CIRCLES

The line joining the points of intersection of two circles is called the common chord. If the equation of two circle.

$S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$, then equation of common chord is S – S' = 0

$\Rightarrow 2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Also the length of the common chord AB is given by $AB = 2\sqrt{a^2 - p^2}$

where 'a' is the radius of one of the given circles and 'p' is the distance of its centre from their common chord.

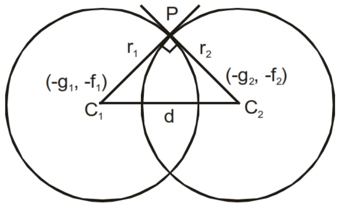
Condition of Orthogonality

If the angle of intersecton of the two circle is a right angle then such circle are called Orthogonal circle. In DPC_1C_2

$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2 \Rightarrow d^2 = r_1^2 + r_2^2$

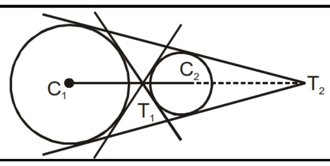
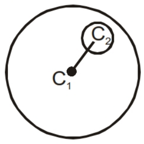
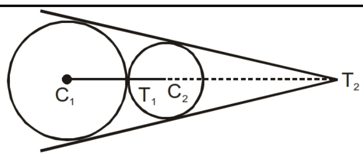
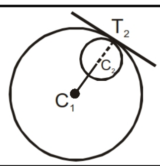
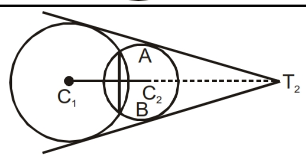
$\Rightarrow \quad (g_1-g_2)^2 + (f_1-f_2)^2 = g_1^{\,2}+f_1^{\,2} - c_1+g_2^{\,2} + f_2^{\,2} - c_2$

$\Rightarrow \quad 2g_1\, g_2 + 2f_1f_2 = c_1 + c_2$ (Condition of Orthogonality)



POSITION OF TWO CIRCLES

Let $C_1\ (h_1\ , k_1)$ and $C_2\ (h_2, k_2)$ be the centre of two circle and r_1, r_2 be their radius then

	CONDITION	POSITION	DIAGRAM	NO. OF COMMON TANGENTS
(i)	$C_1C_2 > r_1 + r_2$	do not intersect or one outside the other		4
(ii)	$C_1C_2 < \|r_1 - r_2\|$	one inside the other		0
(iii)	$C_1C_2 = r_1 + r_2$	external touch		3
(iv)	$C_1C_2 = \|r_1 - r_2\|$	internal touch		1
(v)	$\|r_1 - r_2\| < C_1C_2 < r_1 + r_2$	intersection at two real points		2

Point of intersection of common tangents : The points T_1 and T_2 (points of intersection of indirect and direct common tangents) divide C_1C_2 internally and externally in the ratio $r_1 : r_2$.

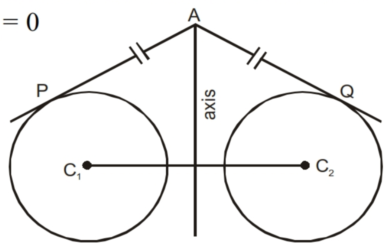
Equation of the common tangent at point of contact : $S_1 - S_2 = 0$

Point of contact : The point of contact C_1C_2 in the ratio $r_1 : r_2$ internally or externally as the case may be.

RADICAL AXIS & RADICAL CENTRE

Radical Axis

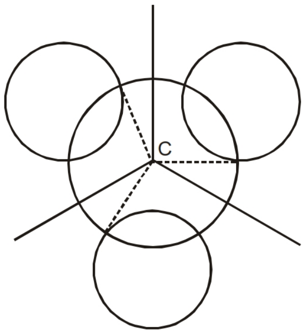
The radical axis of two circle is the locus of a point, which moves in such a way that the lengths of the tangents drawn from it to two given circles are equal. The equation of radical axis of two circle S = 0 and S' = 0 is written as S – S' = 0 i.e., $2(g_1 - g_2)\,x + 2(f_1 - f_2)\,y + c_1 - c_2 = 0$



Note :
- Radical axis biscets every common tangents of two circles.
- Radical axis of two circles is perpendicular to the line joining their centres.
- If two circles intersect a third circle orthogonally, then their radical axis will passes through the centre of third circle.
- Radical axis of three circle, taken two at a time meet at a point provided the centre of the circle are not collinear.
- If two circles intersect at two points then their radical axis coincides with their common chord. Also if they touch each other then it coincides with their common tangent at their point of contact.

Radical Centre

The point where the radical axis of three given circles taken in pairs meet, is called the radical centre of those three circles.



Thus the length of the three tangents drawn from the radical centre on the three circles are equal.

If $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ be any three given circles, then to obtain the radical centre, we solve any two of the following $S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$

Note :
- If the centres of three circles are collinear then their radical centre will not exist.
- The circle with centre at radical centre and radius is equal to the length of tangents from radical centre to any of the circles will cut the three circle orthogonally and is called as Radical circle.
- Circles are drawn on three sides of a triangle as diameter than radical centre of these circles is the orthocentre of the triangle.